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1997 J. Phys. A: Math. Gen. 30 1259

(http://iopscience.iop.org/0305-4470/30/4/026)

J. Phys. A: Math. Gen. **30** (1997) 1259–1271. Printed in the UK PII: S0305-4470(97)78645-1

A realization of quantum algebras—some applications

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Received 8 October 1996

Abstract. Some applications of a new realization of quantum algebras are given. Finite-dimensional $U_q sl(n)$ modules are constructed in a new basis with explicit matrix elements. For q a root of 1 the structure of Weyl modules for $U_q sl(3)$ is analysed in detail. As a corollary, a partial result on tensor product structure is obtained. Generalizations to other quantum algebras are briefly discussed. Realizations of $U_q sl(n)$ in terms of difference operators and some unusual representations are also presented.

A new realization of q-deformed enveloping algebras in a quotient algebra was reported recently [1]. Some elementary applications in the case of $U_q sl(2)$ yielded various representations. A new class of infinite-dimensional representations was also constructed. In this paper, these techniques are extended to more complicated cases. Naturally, one needs more sophisticated tools for such an analysis. The tensor product structure, i.e. the coproduct in the quantum algebras is exploited for this purpose.

Section 1 reviews the realization of $U_q sl(n+1)$ in a quotient algebra Q. Simultaneously, one obtains a realization of $U sl(n+1)$ in Q. Section 2 deals with finite-dimensional representations. Constructions on our new basis are given along with some explicit formulae. In particular, the representations corresponding to fundamental weights are discussed. The non-generic case is covered in section 3 with detailed results in the case of $U_q sl(3) (\simeq U_q su(3))$. A different proof is given of a theorem concerning Weyl modules [2]. As a corollary one obtains partial results on the fusion structure in this case. After a discussion on the advantages of the present realization further applications of the technique are given in section 4. First, an illustration of our method in studying the structure of the tensor product of representations is given. We emphasize that the computations are simpler compared to those using boson realization and also the standard Cartan–Weyl generators. Moreover, the equations for singlar vectors are uncoupled and provide a constructive technique. The second application shows that the q-boson type realization can be obtained as a special case. Section 4 deals with some infinite-dimensional representations. Besides giving a q-derivative realization, some unusual irreducible modules are given. A physical interpretation of these modules in terms of the momentum space representation is given. We conclude this paper with a discussion on possible applications to infinite-dimensional algebras and vertex operator realizations.

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0305-4470/97/041259+13$19.50

1. Realization

Let A be the algebra with unity generated by y_i, h_i, and c where $i \in \{1, \ldots, n\}$ with defining relations $[h_i, y_j] = -a_{ji} y_j$ and all other commutators zero. Here (a_{ij}) is the Cartan matrix of $sl(n+1)$. The following identities are easily verified:

$$h_i^m y_j^n = y_j^n (h_i - n a_{ji})^m. \tag{1}$$

It thus follows that $y_n^{k_n} \ldots y_1^{k_1} h_n^{m_n} \ldots h_1^{m_1} c^r$ form a basis (the PBW basis) for A. Moreover, A has no zero divisors. Let S be the multiplicatively closed set generated by $\{y_i : i = 1, \ldots, n\}$ consisting of the y_i's. Then (1) implies that S satisfies the Ore condition [3]. Thus, there is an algebra of quotients of A with respect to S i.e. a ring P and an injective map $A \longrightarrow P$ such that the image of every element of S is invertible in P and any $p \in P$ can be written as $s^{-1}a$ where $s \in S$ and $a \in A$. Here we identify A with its image in P. Let q be an indeterminate and Q be the appropriate completion of $C[q] \otimes S^{-1}A$ where $C[q]$ is the algebra of rational functions in q. Write formally $q = \exp w$ and consider the following elements in Q

$$\begin{aligned} E_i &= y_i^{-1} \frac{\sinh(w/(n+1)) \left(c + \sum \alpha_{ij} h_j\right)}{\sinh w} \\ F_i &= y_i \frac{\sinh(w/(n+1)) \left(c + \sum \beta_{ij} h_j\right)}{\sinh(w)} \end{aligned} \tag{2}$$

where

$$\begin{aligned} \alpha_{ij} &= (j \leqslant i) = -(n+1-j) \qquad (j > i) \\ \beta_{ij} &= j(j < i) = -(n+1-j) \qquad (j \geqslant i). \end{aligned} \tag{3}$$

Put

$$\begin{aligned} \exp\left(\frac{w}{n+1}\left(c + \sum \alpha_{ij} h_j\right)\right) &= u_i \\ \exp\left(\frac{w}{n+1}\left(c + \sum \beta_{ij} h_j\right)\right) &= v_i. \end{aligned} \tag{4}$$

It follows that

$$u_i y_k^m = q^{m\sigma_{ki}} y_k^m u_i \qquad \sigma_{ki} = \delta_{k,i+1} - \delta_{ki}$$

and

$$v_i y_k^m = q^{m\sigma_{k,i-1}} y_k^m v_i. \tag{5}$$

Here we use the general relation $\operatorname{ad} h(y_i^{-1}) = -y_i^{-1} \operatorname{ad} h(y_i) y_i^{-1}$. Using these formulae one obtains

$$[E_i F_j] = \delta_{ij} \frac{\sinh(w h_i)}{\sinh w}.$$

Moreover, the analogue of Serre-type relations [4] between the F_i's and E_i's are easily verified. Hence, we have a homomorphism from $U_q sl(n+1)$ into Q which yields a morphism $Usl(n+1) \longrightarrow Q$ in the limit $q \to 1$. We thus have a realization of the former. It is to be noted that this is only an algebra morphism. Observe also that in (2) if one interchanges α_{ij} and β_{ij}, respectively, the commutation relations are unaffected. Call this the dual realization and denote it by a prime, i.e. $F_i' = y_i \sinh(w/(n+1) \cdot (c + \sum \alpha_{ij} h_j))$ etc. A useful result is the following.

Lemma 1. In Q

$$F_i^k = y_i^k[v_i; q]_k \qquad \text{and} \qquad (F_i')^k = (-1)^k y_i[u_i^{-1}; q]_k \tag{6}$$

where $[a;q]_n = (a-a^{-1})(aq - a^{-1}q^{-1})\dots(aq^{n-1} - aq^{-n+1})(q-q^{-1})^{-n}$ for $n = 1, 2, \dots$ and $[a;q]_n = 1$ for $n = 0$, and $[n]! = [q;q]_n$ is the (modified) q-shifted factorial. Set $[n] = (q^n - q^{-n})/(q - q^{-1})$.

The proof is an easy consequence of (2) and induction.

It is to be observed that instead of the generators y_i and h_i one could start with y_i and $g_i = \exp(wh_i/n+1) = t^{h_i}$. Then we consider a group G generated by y_i, g_i, t, and μ with defining relations $g_i y_j = t^{-a_{ji}} y_j g_i$. Let K be a field and KG the corresponding group algebra. By identifying q^c with μ (and $t^{n+1} = q$) in (2) one gets a realization of $U_q sl(n+1)$ in a *group algebra*. However, we do not pursue this line of thought in this paper.

2. Representations

Any representation of the algebra A in which the y_i's act as invertible operators will automatically yield a representation of $U_q sl(n+1)$. We assume throughout this section that q is not a root of unity. Consider an A-module N generated by $S \cdot v$, such that $h_i v = \lambda(h_i)v$, where $\lambda \in H^*$, the dual of H generated by the h_i's. We are primarily interested in the highest weight modules. Thus if v is a highest weight vector it must be annihilated by N_+, the subalgebra generated by E_i's. This imposes a condition on the highest weight λ. In fact, by this direct method the $U_q sl(n+1)$ modules constructed have highest weights $k\lambda_1$ or $k\lambda_n$ with λ_i, the ith fundamental weight and k an arbitrary complex number. However, by repeated tensoring one can obtain modules with arbitrary weights. The advantage is a very simple basis and easy computations. Consider first the finite-dimensional representations in the generic case. It is easily seen that the requirement of finite dimensionality implies k must be a non-negative integer.

Let v a be weight vector with weight $k\lambda_1$, i.e. $h \cdot v = k\lambda_1(h) \cdot v$. Let the extended Q-module $M = \{y_n^{k_n} \dots y_1^{k_1} \cdot v \mid k_i \in Z\}$. Let the generators of $U_q sl(n+1)$ be given by the realization (2) such that the central element c act as the scalar $-k$. Thus one has $E_i \cdot v = 0$, $i = 1 \dots n$. Moreover, using formula (2) it follows that

$$\begin{aligned} E_i y_n^{k_n} \dots y_1^{k_1} \cdot v &= [k_{i+1} - k_i] y_n^{k_n} \dots y_i^{k_i - 1} \dots y_1^{k_1} \cdot v \\ F_i y_n^{k_n} \dots y_1^{k_1} \cdot v &= [k_i - k_{i-1} - k\delta_{i1}] y_n^{k_n} \dots y_i^{k_i} \dots y_1^{k_1} \cdot v. \end{aligned} \tag{7}$$

Similarly for the dual module corresponding to the highest weight $k\lambda_n$ one has $c = k$ and

$$\begin{aligned} E_i' y_n^{k_n} \dots y_1^{k_1} \cdot v &= [k_i - k_{i-1}] y_n \dots y_i^{k_{i+1}} \dots y_1^{k_1} \cdot v' \\ F_i' y_n^{k_n} \dots y_1^{k_1} \cdot v &= [k_{i+1} - k_i + k\delta_{in}] y_n^{k_n} \dots y_i^{k_{i+1}} \dots y_1^{k_1} \cdot v. \end{aligned} \tag{8}$$

The above formulae enable one to read off the singular vectors, i.e. all $u \in M$ s.t. $E_i u = 0$ (or $F_i u = 0$) for $i = 1, \dots, n$. Thus, in the case of $k\lambda_1$, the only E-singular vector is v and the F-singular vector is $y_n^k \dots y_1^k \cdot v$. Therefore, the subspace V of M spanned by $y_n^{k_n} \dots y_1^{k_1} \cdot v$, with $0 \leqslant k_n \leqslant \dots \leqslant k_1 \leqslant k$, is in fact an $U_q sl(n+1)$ submodule. Since it is finite-dimensional with an essentially unique E-singular vector it is irreducible. Similarly, $V(k\lambda_n)$ is spanned by $y_n^{k_n} \dots y_1^{k_1} \cdot v$ with $k \geqslant k_n \geqslant \dots \geqslant k_1 \geqslant 0$. The formulae (7) provide the matrix elements. Moreover, we have a natural pairing corresponding to the transpositions $i \leftrightarrow n+1-i$. It may be noted that with some modification the above bases correspond to the crystal bases of Kashiwara [5] for the modules concerned.

Next we consider modules corresponding to other fundamental weights. As mentioned earlier these can be constructed by appropriate tensor products. One has to use the coproduct structure in $U_q sl(n+1)$. There is a coproduct in Q. The relation between these two coproducts, if any, should be interesting. But we use only the well known coproduct in $U_q sl(n+1)$. These are given by, setting $K_i = q^{h_i}$,

$$\triangle(h) = h \otimes 1 + 1 \otimes h \qquad h \in H \tag{9}$$

$$\triangle(E_i) = E_i \otimes 1 + K_i \otimes E_i \tag{10}$$

and

$$\triangle(F_i) = F_i \otimes K_i + 1 \otimes F_i. \tag{11}$$

These formulae define the action of the generators on $M \otimes N$ where M and N are $U_q sl(n+1)$ modules. This is further extended to an arbitrary but finite number of tensor products.

Now, consider the module $V = V(\lambda_1)$ (i.e. the regular representation). Let $E = \sum_{k=0}^{n} \bigwedge^k(V)$ denote the sum of the kth exterior powers of V. As in the case of $sl(n+1)$ the 'k-vectors', $v \wedge y_1 v \wedge \cdots \wedge y_{k-1} \ldots y_1 v$ generate the module $V(\lambda_k)$. In fact, the explicit formulae can be written down with ease. Thus, let $v_{(i)} = y_{i-1} \ldots y_1 v$. The k-vectors $v_{(i_1)} \wedge \cdots \wedge v_{(i_k)}$, $i_1 < \cdots < i_k$, span $V(\lambda_k)$. One sees easily that the matrix elements in this basis are identical to those in the ordinary case $(q \to 1)$. From the modules $V(\lambda_i)$ one constructs $V(k_i \lambda_i)$ by taking symmetric powers and then computes the matrix elements for any $V(\lambda)$. The calculations are simple but tedious. The details are given only for $U_q sl(3)$ below. It may be noted that various aspects of this case have been investigated by other methods [6]. However, the present method besides covering these aspects extends to the difficult case when q is a root of 1.

The algebra $U_q sl(3)$ is special in the $sl(n+1)$ series. The only two fundamental modules are the regular representation and its dual. Therefore, from the construction above any module $V(\lambda)$ with $\lambda = k_1\lambda_1 + k_2\lambda_2$, $k_1, k_2 \geqslant 0$, is easily constructed. For this we have to use the coalgebra structure in $U_q sl(3)$. The following lemma is well known [6].

Lemma 2.

$$\triangle(E_i^n) = (\triangle(E_i))^n = \sum_{0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n(n-k)} E_i^k K_i^{n-k} \otimes E_i^{n-k}$$

$$\triangle(F_i)^n = \sum \begin{bmatrix} n \\ k \end{bmatrix} q^{n(n-k)} F_i^k \otimes F_i^{n-k} K_i^{-k} \tag{12}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

This is an analogue of the binomial theorem for q-commuting variables and easily proved by induction.

Let $V(j\lambda_1)$ be the $U_q sl(3)$-module with generators by E_i and F_i, $i = 1, 2$. The action of these generators is given by (7). Similarly, $V(k\lambda_2)$ is defined via the action of E_i' and F_i'. If x and z are the respective highest weight vectors then bases for $V(j\lambda_1)$ and $V(k\lambda_2)$ are given by $y_1^{m_1} y_2^{m_2} \cdot x$ and $y_1^{n_1} y_2^{n_2} \cdot z$, with $m_1 \geqslant m_2$ and $n_2 \geqslant n_1$. Let $\lambda = j\lambda_1 + k\lambda_2$ and $W = V(j\lambda_1) + V(k\lambda_2)$. The latter is reducible but one finds the indecomposable components, for example the irreducible submodule $V(\lambda)$ generated by $x \otimes z$. For q generic, W is completely reducible and $V(\lambda)$ is irreducible [7]. In general, neither of these statements is true in the non-generic case (q is a root of 1). In any case, the

Weyl modules $V(\lambda)$ are of fundamental importnace. Let $v = x \otimes z$ be the highest weight singular vector that generates $V(\lambda)$. The usual basis for $V(\lambda)$ is $F_3^{m_3} F_2^{m_2} F_1^{m_1} \cdot v$, where $F_3 = F_1 F_2 - q^{-1} F_2 F_1$. Consider now the relation between the two bases discussed here.

Lemma 3. The following identity holds in $U_q sl(3)$:

$$F_2^n F_3^m (x \otimes z) = q^{-jr} \sum_r \begin{bmatrix} m \\ r \end{bmatrix} q^{m(m-r)} F_3^r \cdot x \otimes F_2^n F_3^{m-r} \cdot z. \tag{13}$$

Proof. Note that $\triangle(F_3) = F_3 \otimes q^{-h_3} + 1 \otimes F_3 + (q - q^{-1}) F_2 \otimes q^{-h_2} F_1$. Since $F_2 \cdot x = F_1 \cdot z = 0$ and $\triangle(f) \cdot (x \otimes z) = F_3 \cdot x \otimes q_{-h^3} \cdot z + 1 \otimes F_3 \cdot z$ the above identity follows from the q-binomial theorem and the fact that $F_2 F_3 = q^{-1} F_3 F_2$. □

Theorem 1. Let x and z be the highest weight vectors in $V(j\lambda_1)$ and $V(k\lambda_2)$, respectively. The actions of the generators of $U_q sl(3)$ on the respective modules are given by (7) and (8) (for $n = 2$). Then

$$F_1^c F_2^b F_3^a (x \otimes z) = \sum_{r,s} \alpha q^t \begin{bmatrix} a \\ r \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix} [q^{-j+r}; q]_s [q^b; q^{-1}]_{c-s} [q^{k-a+r}; q]_{c-s} [q^{k-a+r}; q^{-1}]_b$$

$$\times [q^k; q^{-1}]_{a-r} y_1^{r+s} y_2^r \cdot x \otimes y_1^{a+c-r-s} y_2^{a+b-r} \cdot z \tag{14}$$

where

$$t = -(k+1)r + (a - r - b)s + (a - r)r + (c - s)s$$

and

$$\alpha = (q - q^{-1})^{a+b+c}.$$

This result follows from the preceding lemma and the formulae (7) and (12).

Next consider the question of reducibility. A Verma module is by definition generated by a singular vector with highest weight that is annihilated by $E_i (i = 1, 2)$. Verma modules are indecomposable. The Weyl modules $V(\lambda)$ are quotients of Verma modules. Both in the cases of ordinary Lie algebras and their q-analogues when q is not a root unity $V(\lambda)$ is irreducible. To prove irreducibility in the case of $V(\lambda)$ which is finite-dimensional it is necessary and sufficient that there be no singular vector other than the highest weight vector. Thus it is essential to find the singular vectors [8] in any finite-dimensional module. In this direction we have:

Lemma 4. In the $U_q sl(3)$-module, $V(j\lambda_1) \otimes V(k\lambda_2)$, any singular vector is of the form $\sum_{m,n} C_{mn} y_2^m y_1^n \cdot x \otimes y_2^{M-m} y_1^{N-n} \cdot z$ where C_{mn} satisfy the recursion relations:

$$C_{m,n+1} = \frac{q^{N-n} - q^{-N+n}}{q^{n-m+1} - q^{-n+m-1}} q^{j+m-2n} C_{mn} \tag{15}$$

$$C_{m+1,n} = -\frac{q^{M-N+n-m} - q^{-(M-N+n-m)}}{q^{m+1} - q^{-m-1}} q^{n-2m} C_{mn}. \tag{16}$$

First, one notes that any singular vector must belong to some weight space of the Abelian subalgebra generated by h_i's. Thus it must have the form given above. The recursion relations follow from direct computation.

Next note that if q is the pth root of unity then the module $V(j\lambda_1)$ is reducible for $j \geqslant p/2, p$ for even and odd p, respectively. For definiteness we consider only the restricted modules in the non-generic case, i.e. $j, k < p$; p odd. We can deal with the case of even p and non-restricted modules with a little more effort.

Theorem 2. Suppose q is generic or $j, k < p$ if q is a pth root of unity, p odd. Then any singular vector v in $V(j\lambda_1) \otimes V(k\lambda_2)$ is given by the formula (up to a scalar multiple)

$$v = \sum_{m,n} C_{mn} y_2^m y_1^n x \otimes y_2^{N-m} y_1^{N-n} z$$

$$C_{mn} = \frac{[N]!}{[N-n]![m]![n-m]!} q^{n(j+1-n)+m(n+1-m)} \qquad N \leqslant \min(j,k). \quad (17)$$

Proof. Equation (15) implies that the minimum value of n is m and the maximum N. Thus $\max(m) = n$ and then (16) implies $M = N$. Furthermore, $\min(m) = 0$. Since, $0 \leqslant m \leqslant n \leqslant j$ and $N - n \leqslant N - m \leqslant k$ we get $N \leqslant \min(j,k)$. The coefficients C_{mn} are then easily computed from the recursion relations (15) and (16). □

We note the simple form of the singular vectors in our basis. The theorem implies that the singular vectors belong to the subspaces with weights $\lambda = j\lambda_1 + k\lambda_2, (j-1)\lambda_1 + (k-1)\lambda_2, \ldots, (k-j)\lambda_1$ assuming $j \leqslant k$. Now consider the submodule $V(\lambda)$ generated by $x \otimes z$, with highest weight $j\lambda_1 + k\lambda_2$. For generic q this is the only singular vector in this submodule. One can infer this from the general theory of representations of quantum algebra which is parallel to the ordinary case ($q = 1$). However, we can prove this directly by showing that the singular vectors in (17) cannot belong to the submodule spanned by the vectors $F_1^c F_2^b F_3^a (x \otimes z)$. The second course becomes imperative in case q is a root of unity since the classical theorems such as the Weyl reducibility theorem and tensor product decomposition theorems are not true. We mention in passing that one faces a similar situation in the case of classical Lie algebras over a field of prime characteristics (modular Lie algebras) [9]. The method given in the following section for quantum algebras can easily be adapted to modular Lie algebras.

3. Root of 1 case

Throughout this section it is assumed that q is a pth root of unity, p odd. As mentioned earlier, extension to the even case is straightforward with minor changes. Now let v be a singular vector in $V(j\lambda_1) \otimes V(k\lambda_2)$ given by the formula (17). Comparing (14) and (17) one proves the following.

Lemma 5. If a singular vector $v \in V(\lambda)$ then it has the form

$$v = \sum_i d_i F_1^i F_2^i F_3^{N-i} (x \otimes z) \qquad 0 < N < \min(j,k). \quad (18)$$

Here the coefficients d_i are to be determined. Translating into our basis we obtain

$$\sum_{m,n} C_{mn} y_2^m y_1^n x \otimes y_2^{N-m} y_1^{N-n} z = \sum_{i,r,s} d_i F_i(r,s) \alpha_i q^{a_i} (y_1^{r+s} y_2^r x \otimes y_1^{N-r-s} y_2^{N-r} z) \quad (19)$$

$$F_i(r,s) = \frac{(-1)^{r+s}([i]!)^2 [N-i]![j]![k]!}{[s]![i-s]![r]![N-i-r]![s]![k+r-N]![j-r-s]!} \quad (20)$$

$$a_i = -(k+1)r + (N-i-(r+s))s + (N-(i+r))r, \alpha_i = (q-q^{-1})^{i-N}.$$

Now, putting $r = m$ and $r + s = n$ in (19) one obtains a set of equations

$$C_{mn} = \sum_{l=0}^{N-n} d_{n-m+l} \alpha_{n-m+l} q^{a_{n-m+l}} f_{n-m+l}(m, n-m). \quad (21)$$

Any singular vector v given by (18) must satisfy these $(N+1)(N+2)/2$ equations. Thus our task is to determine the conditions under which (21) has solutions and to calculate the coefficients d_i. Assuming (21) is satisfied we can calculate the coefficients d_i:

$$C_{NN} = (-1)^N d_0 q^{-(k+1)N} \frac{[j]!}{[j-N]!} \tag{22}$$

$$C_{N-1,N} = (-1)^N d_1 q^{-(k+1)(N-1)-1} \frac{[j]![k]!}{[j-1]![k-N]!} \tag{23}$$

$$C_{N-i,N} = (-1)^N d_i q^{-(k+1)(N-i)-i^2} \frac{[j]![k]!}{[k-i]![j-N]!}. \tag{24}$$

Furthermore, from (21) we get

$$\begin{aligned} C_{N-1,N-1} = {} & (-1)^{N-1} q^{-k(N-1)} \frac{[N]![j]![k]!}{[N-1]![k-1]![j-N+1]!} d_0 \\ & + q^{-(k+1)N-1} \frac{[j]![k]!}{[k-1]![j-N+!]!} d_1. \end{aligned} \tag{25}$$

Using the preceding formulae this reduces to

$$\frac{C_{N-1,N-1}}{C_{N-1,N}} = -\frac{[N]![j-N]![k]!}{[N-1]![j-N+1]![k-1]!} \frac{C_{NN}}{C_{N-1,N}} - \frac{q[j-N]!}{[j-N+1]!}. \tag{26}$$

Now using (15) and (16) the above simplifies to

$$q^{-(j-N+1)} = -\frac{[j-N]!}{[j-N+1]!} \left(\frac{[k]!}{[k-1]!} q^{k+2} + q \right). \tag{27}$$

If q is not a root of unity (27) can be satisfied if and only if $j+k = N-2$ which is impossible since $N < \min(j,k)$. Therefore, we have a direct proof of irreducibility of $V(\lambda)$ in the generic case. If q is a pth root of unity (p odd) then (27) is satisfied iff

$$j + k = p + N - 2. \tag{28}$$

Now we can prove the main theorem.

Theorem 3. Let $V(\lambda = j\lambda_1 + k\lambda_2)$ be the Weyl module generated for $U_q sl(3)$ by a highest weight singular vector v. If q is generic $V(\lambda)$ is irreducible. If q is a pth (p odd) root of 1 then the following statements are true: (i) $V(\lambda)$ is irreducible when either $j+k < p-1$ or $j = p-1$ or $k = p-1$; (ii) if $j+k = p+N-2$, $N > 0$, $j, k < p-1$ then $V(\lambda)$ is reducible. The irreducible submodule $V(\lambda')$ of $V(\lambda)$ is of highest weight $\lambda' = (j-N)\lambda_1 + (k-N)\lambda_2$. The corresponding singular vector v' is given by the following formula:

$$v' = \sum_{i=0}^{N} d_i F_1^i F_2^i F_3^{N-i} \cdot v \tag{29}$$

$$d_i = \frac{[k-i]!}{[N-i]![i]!} q^{ji}. \tag{30}$$

The quotient module $V(\lambda)/V(\lambda')$ is irreducible and its dimension is equal to the difference of dimensions of the corresponding modules given by the Weyl dimension formula for simple Lie algebras over $\mathbb{C}$.

Proof. The first assertion has already been proved except for the case j or $k = p - 1$. However, this is easy to see since (28) is satisfied for $N = j + 1$ (assuming $k = p - 1$) and by theorem 2 all singular vectors satisfy $N < \min(j, k)$. To prove (ii) note first that the Weyl module $V(\lambda)$ is by definition obtained from the corresponding irreducible module by appropriate specialization in $\mathbb{C}[q]$ [7]. One can carry out these steps because of the existence of a 'Z-basis' in $U_q L$ for any simple Lie algebra L. Explicitly, let ϵ be root of unity and let K be the image of the homomorphism $\mathbb{C}[q] \longrightarrow \mathbb{C}[\epsilon]$ sending $q \to \epsilon$. Let $V_\epsilon(\lambda) = V(\lambda) \bigotimes_K \mathbb{C}$ where $V(\lambda)$ is the highest weight $U_q L$ module when q is a formal variable. Then $V_\epsilon(\lambda)$ is the Weyl module corresponding to λ. We drop the formal notation and treat q as a root of unity and note that the Weyl module is isomorphic to the tensor product modules considered above. Hence, to show that v' given by (29) is a singular vector it suffices to show that d_i satisfy the equations (21). Note that if d_i satisfy (21) then they are determined uniquely up to a constant (24). In fact, (29) follows (24) and (17). Substituting the values of d_i in (21) and simplifying we have to prove the following identity

$$\frac{q^{n(j+1-N)+m(k+2)}}{[N-n]!} = \alpha(-)^n \sum_{r=0}^{N-n} \frac{q^{(j-n)(r+n-m)}[k-s-r]![s+r]!}{[r]![N-n-r]![k+m-N]![s]![j-n]!} \tag{31}$$

where $s = n - m$ and α is a constant independent of n and m. For proving (31) we require the following.

Lemma 6. Let q be a pth root of unity and suppose it is given that $A, B \in Z_+$ be such that $A, B \leqslant p - 1$ and $A + B = p + t$, then for $h \leqslant t$

$$\sum_{k=0}^{h} \frac{[A]![B]!}{[k]![A-k]![h-k]![B-h+k]!} q^{-Ah+tk} = \frac{[t]!}{[t-h]![h]!}. \tag{32}$$

Formula (32) is simply an adaptation of a well known relation among Gaussian binomial coefficients [10] to our notation. On the right-hand side, t instead of $A + B$ appears due to the fact that q is a root of unity. Furthermore, defining for any integer $i \leqslant p - 1$, $i' = p - 1 - i$ we have

$$[i]! = (-1)^i \frac{p}{[i']!}. \tag{33}$$

Then apart from an irrelevant factor the right-hand side of (31) is equal to

$$\frac{q^{(j-n)(n-m)}}{[j-n]!} \cdot \sum_{r=0}^{N-n} \frac{[s]![j-m+1]!}{[s-r]![r]![j-N+s+r+1]![N-n-r]!} q^{r(j-n)}.$$

Now using the lemma and the fact that $s = n - m$ it is easily seen that apart from a factor independent of m and n the last expression is equal to the left-hand side of (31). The theorem is proved. □

It may be observed that with a little more effort the general non-restricted case, i.e. allowing $k \geqslant p$ may be treated by this method. In the case of even p replace p by $p/2$ everywhere in the theorem above. Note also that the case of ordinary Lie algebras over a field of characteristic p has a parallel representation theory. In fact, a part of the theorem has been proved for arbitrary $U_q L$ by different methods [2]. However, the present method yields more viz an explicit construction of singular vectors in terms of two bases—the standard one and a polynomial basis given here. Furthermore, an analysis parallel to the one above would also yield the F_i-singular vectors (i.e. those annihilated by F_i) which can also be derived by using an appropriate involution operater on e-singular vectors. Using these one can build a picture of indecomposable $U_q sl(3)$ modules. Besides, theorem 3 along with the results preceding it can be used to obtain a tensor product theorem for modules of type $V(j\lambda_1)$ and $V(k\lambda_2)$. Thus we have the following.

Corollary 1. Using the notation of the theorem and $\delta = \alpha_1 + \alpha_2$ where α_1 and α_2 form the root basis of $sl(3)$ and $j \leqslant k < p - 1$

$$V(j\lambda) \otimes V(k\lambda) = W_1 \oplus W_2$$
$$W_1 = V(\lambda) \oplus V(\lambda - \delta) \oplus \cdots \oplus V(\lambda - N'\delta)$$

$N' = (N-1)/2$, $(N-2)/2$ for odd and even N, respectively, and

$$W_2 = V(\lambda - (N+1)\delta) \oplus V(\lambda - (N+2)\delta) \oplus \cdots \oplus V(\lambda - j\delta) \oplus X$$

where $X = 0$ for N odd and $X = V(\lambda - (N/2)\delta)$ for even N. W_1 is completely reducible and W_2 is a direct sum of indecomposable modules described in the theorem.

There are some papers [11] which use established techniques (Gelfand–Tsetlin construction, q-boson realization etc) and partially overlap the matters discussed here. Therefore, it seems appropriate to make some comparisons of the present techniques with the standard techniques. First, this is a realization in the semidirect product of two power series algebras. The computations are easier. Second, the calculation of singular vectors is more managable because the recursion relations such as (15) and (16) are uncoupled in contrast to the direct methods using the Cartan–Weyl basis where a consistency requirement is imposed. We illustrate this fact later. From the singular vectors others vectors can be constructed using, for example, formulae such as (14). Moreover, the explicit construction of a basis can be generalized to higher algebras. Thus the present realization can be used as a supplement to the Gelfand–Tsetlin technique and one would expect that the combination of the two would yield rich dividends. Finally, we can embed other quantum algebras in an appropriate quotient algebra. Examples of A_n, and C_n are given later.

4. Further generalizations

First consider the singular vectors in the module $V(k_1\lambda_1) \otimes V(k_2\lambda_2) \otimes V(j\lambda_1)$. A typical homogeneous element is of the form

$$\sum C_{m_1 m_2 n_1 n_2} (y_1^{m_1} y_2^{m_2} u \otimes y_1^{n_1} y_2^{n_2} v \otimes y_1^{N_1 - m_1 - n_1} y_2^{N_2 - m_2 - n_2}) w$$

where u, v, and w are the respective primitive generators and N_1, N_2 are fixed. This is a singular vector if and only if the coefficients satisfy the following set of relations:

$$[m_2 - m_1 - 1] C_{m_1+1 m_2 n_1 n_2} + q^{\lambda_1 - 2m_1 + m_2} [n_1 + 1] C_{m_1 m_2 n_1 + 1 n_2}$$
$$= -q^{\lambda_1 - 2m1 + m_2 - 2n_1 + n_2} [N_2 - m_2 - n_2 - N_1 + m_1 + n_1] C_{m_1 m_2 n_1 n_2}$$

and

$$-[m_2 + 1] C_{m_1 m_2 + 1 n_1 n_2} + q^{-2m_2 + m_1} [n_2 - n_1 + 1] C_{m_1 m_2 n_1 n_2 + 1}$$
$$= q^{\lambda_2 - 2m_2 + m_1 - 2n_2 + n_1} [N_2 - m_2 - n_2] C_{m_1 m_2 n_1 n_2}.$$

From the structure of the modules we know that $m_2 \leqslant m_1 \leqslant k_1$ and $n_1 \leqslant n_2 \leqslant k_2$. Thus the minimum value for m_1 is m_2 and for n_2 it is n_1. We can also deduce that for fixed $m_2, n_2 \max(m_1) = N_1 - N_2 + m_2$ and for fixed $m_1, n_1 \max(n_2) = N_2 - m_1$. Moreover, $(N_1 - N_2) \geqslant (m_1 - m_2) - (n_1 - n_2)$. In particular, $N_1 \geqslant N_2$. Note that the two sets of equations are not coupled in contrast to the standard technique with the Cartan–Weyl basis. We also have $\min(m_1+n_1) = N_1 - j$ and $\min(m_2+n_2) = N_2 - j$. Next note that for the coefficient of the form C_{r+krss} the first equation reduces to

$$C_{r+k+1rss} = -q^{\lambda_1 - r - s - 2k} \frac{[N_1 - N_2 - 2r - k + 2s]}{[k+1]} C_{r+krss}.$$

Thus

$$C_{r+krss} = (-1)^k q^{(\lambda_1 - r - s - k)(k-1)} [N_1 - N_2 - 2r - 2s; q]([k]!)^{-1} C_{rrss}. \qquad (34)$$

Similarly,

$$C_{rrss+k} = q^{(\lambda_2 - s - k)(k-1)}([k]!)^{-1}[n_2 - 2s; q^{-1}] C_{rrss}. \qquad (35)$$

It is seen that $C_{0011}, C_{0022}, \ldots, C_{00N_2N_2}$ already determine the rest of C_{rrss}. Moreover, one shows easily that $0 \leqslant N_1 - N_2 \leqslant j$ and letting $N_1 - N_2 = r$ we have $r \leqslant N_1 \leqslant j + k_1 - r$ and $0 \leqslant N_2 \leqslant k_2 + r$. The last two follow because the highest weight must be dominant. Since the coefficients C_{00kk}, $k \leqslant N_2$, can be chosen independently the number of times a particular module corresponding to a fixed (N_1, N_2) appears in the product $V(k_1\lambda_1) \otimes V_{(k_2\lambda_2)} \otimes_V (j\lambda_1)$ is equal to $\min(N_2+1, N_1+1)$. The coefficients $C_{m_1m_2n_1n_2}$ can be computed from the above relations. These yield all singular vectors when q is the root of 1. But of course, in the latter case there is a lot of collapsing as we have seen in theorem 3. Again the method used in proving the product structure may be applied in this case. However, the computations are involved and it seems likely that some summation formula involving basic hypergeometric series will come in to play again (the simplest such formula is the q-Chu–Vandermonde formula for the series used in theorem 3).

Next using the coproduct structure we give a realization in the case of $U_q sl(n)$ which yields a polynomial basis for all finite-dimensional modules. Let $B = \bigoplus_1^n A^{(i)}$ be the direct sum of m copies of A defined in section 2. Write $x^{(i)}$ for an element in $A^{(i)}$ and consider the following realization

$$E_i = \sum_{j=1}^{m} \prod_{r=1}^{j-1} K_i^{(r)} y_i^{(j)} (w_i^{(j)} - (w_i^{(j)})^{-1}) \qquad (36)$$

where $w_i^{(j)}$ is either u_i or v_i. This can yield all finite-dimensional representations on factor rings of appropriate polynomial rings.

We end this section with formulae giving a related realization of quantum algebras corresponding to classical Lie algebras A_n and C_n. Note that if we put

$$N_i = \sum_j \alpha_{i-1j} h_j = \sum \beta_{ij} h_j$$

then $F_i = y_i[N_{i+1}]$, $E_i = y_i^{-1}[N_i]$, and $K_i = q^{N_{i+1}-N_i}$ for $1 \leqslant i \leqslant n$ gives a realization of $U_q sl(n)$. We could start with generators y_i, N_i and defining relations

$$[N_i y_k] = (\delta_{ik} - \delta_{ik-1}) y_k. \qquad (37)$$

Note that we have $(n+1)$ generators N_i. Let z_i, $i = 1, \ldots, n$, be defined such that

$$[N_i z_j] = \delta_{ij} z_j. \qquad (38)$$

Then setting

$$y_i = z_i z_{i+1}^{-1} \qquad (39)$$

we obtain a realization of (37). Now using these new generators we can get realizations of the quantum algebras of respective classical algebras as follows.

U_qA_n

$$F_i = z_i z_{i+1}^{-1}[N_{i+1}]$$
$$E_i = z_{i+1}z_i^{-1}[N_i]$$
$$K_i = q^{N_{i+1}-N_i} \tag{40}$$

U_qC_n

$$F_i = z_i z_{i+1}^{-1}[N_{i+1}]$$
$$E_i = z_{i+1}z_i^{-1}[N_i]$$
$$K_i = q^{(N_{i+1}-N_i)} \qquad i = 1, \ldots, n-1$$
$$F_n = z_n^2$$
$$E_n = -z_n^{-2}\begin{bmatrix} N_n \\ 2 \end{bmatrix}$$
$$K_n = -[2N_n + 1]. \tag{41}$$

We note that, unlike the undeformed case, $B_q(n)$ and $D_q(n)$ have realization only in Fermionic generators [12]. Therefore, although we can obtain realization of B_n and D_n in our generators it is not possible to extend this directly to the deformed case, the reason being that in the latter case the coproduct is not cocommutative.

5. Operator realizations

In this section we construct some infinite-dimensional representations. First, we have a realization of the algebra A in terms of differential operators. See [13] for a similar construction for $U_qsl(2)$. Let $B = \mathbb{C}[z_i, z_i^{-1}; i = 1, \ldots, n]$ be the algebra of Laurent polynominals in z, treated here as a formal power series. Let D denote the derivation $z(\partial/\partial z_i)$ and set $D_i = 0$ if $i = 0$ or $n+1$. Let

$$h_i = D_{i-1} - 2D_i + D_{i+1} \tag{42}$$

and

$$y_i = z_i. \tag{43}$$

Then $[h_i y_j] = -\alpha_{ji} y_i$. Therefore, using (2) we obtain the realization (putting $c = 0$):

$$E_i = z_i^{-1}\frac{q^{D_{i+1}-D_i} - q^{D_i - D_{i+1}}}{q - q^{-1}}$$
$$F_i = z_i \frac{q^{D_i - D_{i-1}} - q^{D_{i-1}-D_i}}{q-q^{-1}}. \tag{44}$$

In terms of the shift operators $T_i \cdot f(z_1, \ldots, z_i, \ldots) = f(z_1, \ldots, qz_i, z_{i+1}, \ldots)$

$$E_i = z_i^{-1}\frac{T_{i+1}T_i^{-1} - T_i T_{i+1}^{-1}}{q-q^{-1}}$$
$$F_i = z_i \frac{T_i T_{i-1}^{-1} - T_i^{-1}T_{i-1}}{q - q^{-1}}. \tag{45}$$

One could also write the above in terms of partial q-derivatives but (45) is more useful. Let us briefly consider a possible method of constructing an infinite-dimensional representation. Let A be an Abelian algebra with generators $\{\alpha_i\}$ and suppose that A has no zero divisors. Let $B \subset \mathrm{Der}(A)$ be an Abelian subalgebra of $\mathrm{Der}(A)$, the algebra of derivations of A.

Let S be the subalgebra of A generated by simultaneous eigenvectors of B. Then, setting $h_i = \sum j d_j^{(i)}$, $d_j^{(i)} \in B$, and choosing $y_k \in S$ we get a set of linear equations such that $[h_i y_j] = -a_{ji} y$. Then using (2) one obtains a realizaton of $U_q sl(n+1)$. Note that we could slightly generalize this by putting $h = k + D$, k constant, and choosing k appropriately. One could then calculate n-point functions for the modules [13].

Finally, let us consider an unusual representation corresponding to Whittaker modules [1, 14]. Thus, let $X = \{f(z_1, \ldots, z_n): f$ is meromorphic in each variable$\}$. The domains of definitions of $f \in X$ will be left unspecified. Let $z = (z_1, \ldots, z_n)$

$$h_i f(z) = z_i f(z)$$
$$y_i \cdot f(z) = f(z_1, \ldots, z_{i-1} - 1, z_i + 2, z_{i+1} - 1, \ldots, z_n).$$

Then, h_i and y_j satisfy the basic commutation relations. Note that h_i is no longer diagonal. In terms of operators

$$E_i = q^{(\sum \alpha_{ij} z_j - 1)} - q^{-(\sum \alpha_{ij} z_j - 1)}$$
$$= \frac{\exp(\partial_{i-1} - 2\partial_i + \partial_{i+1})}{q - q^{-1}}.$$

Still another similar non-standard realization of $U_q sl(n+1)$ is given by defining

$$h_i f(z) = z_i f(z)$$
$$y_i f(z) = \frac{f(z_1, \ldots, z_{i-1} - 1, z_i + 2, z_{i+1} + 1, \ldots, z_n)}{(z_{i-1} - 1)(z_i + 2)(z_{i+1} - 1)}. \tag{46}$$

Therefore,

$$E_i = \frac{q^{\sum \alpha_{ij} z_j - 1} - q^{-\sum \alpha_{ij} z_j - 1}}{(z_{i-1} + 1)(z_i - 2)(z_i + 2)(q - q^{-1})} \exp(-\partial_{i-1} + 2\partial_i - \partial_{i+1}). \tag{47}$$

Similarly for F_i.

We can interpret these realizations as follows. Let us start with a representation of A in coordinate space:

$$h_k = -\iota \partial_k \tag{48}$$
$$y_k = \exp \iota (z_{k-1} - 2z_k + z_{k+1}). \tag{49}$$

The momentum space representations of these operators via Fourier transforms and their complex cojugates are precisely the representations given above.

6. Conclusion

Some concluding remarks are in order. It is to be observed that the algebra $Q = S^{-1}A$ is isomorphic to a subalgebra of the enveloping field of a Heisenberg algebra [15]. However, note that we have to find a representation of A such that the y_i's are invertible to yield a representation of $sl(n+1)$ and $U_q sl(n+1)$. The representations are on the space of polynomials. We have already noted the relative merits and limitations of this realization. Using this one can get all the results that are computable using the Cartan–Weyl generators or the q-boson realization. Moreover, as discussed at the end of section 3 in the more complicated cases the present method demands relatively easier computations. In section 4 we have obtained some results on the fusion structures in $U_q sl(3)$ modules. This could be a starting point for more complete results with possible applications to quantum field theories with non-semisimple gauge symmetry [16]. Note also that the realization (2) yields a realization of $U_q A(\infty)$ provided we replace the n in those formulae by any constant

($\neq -1$). However, we can pass over to the affine algebras $sl(n)^\wedge$ case. It is also possible to construct vertex operator realizations starting from the present realization.

Acknowledgments

The author is grateful to Professor M Jimbo for his hospitality and many illuminating discussions. He is also thankful to Professor K C Tripathy for useful discussions and Professor H D Doebner for inspiration.

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